Math 254-1 Exam 3 Solutions

1. Carefully state the definition of "dimension", in the context of this course. Give two examples: a four-dimensional vector space, and an infinite-dimensional vector space.

    The dimension of a vector space is the number of vectors in any basis. The most familiar four-dimensional vector space is, of course $\mathbb{R}^4$; but also we have seen $\mathbb{R}_3[t]$, the set of polynomials of degree of at most three. $\mathbb{R}[t]$, the set of all polynomials, is infinite dimensional, as is C^0, the set of continuous functions.

2. Suppose that A, B are square, $n \times n$, invertible matrices. Prove that AB is invertible, and that $(AB)^{-1} = B^{-1}A^{-1}$.

    We calculate $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. Hence AB is invertible, and its inverse is $(B^{-1}A^{-1})$.

The remaining problems all concern the following matrix: $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

3. Be sure to justify your answers to the following questions.

    (a) Is A diagonal? (b) Is A triangular? (c) Is A orthogonal? (d) Calculate $\mathrm{tr}(A)$. (e) Calculate A^T.

    (a) Diagonal matrices are zero for all entries $(B)_{ij}$ with $i \neq j$. A has four nonzero entries off the diagonal, hence is *NOT* diagonal.
    (b) Triangular matrices come in two types: upper triangular matrices are zero for $(B)_{ij}$ with $i > j$; lower triangular matrices are zero for $(B)_{ij}$ with $i < j$. A is not of either type, since $(A)_{13} = (A)_{31} = 1$, hence is *NOT* triangular.
    (c) Orthogonal matrices B satisfy $BB^T = I$; however $AA^T = \begin{bmatrix} 6 & 8 & 1 \\ 8 & 20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \neq I$. Hence A is *NOT* orthogonal.
    (d) The trace of a matrix is the sum of its diagonal entries, in this case $1+2+0 = 3$.
    (e) The transpose of a matrix is calculated by swapping $(B)_{ij}$ with $(B)_{ji}$; hence $A^T = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

4. Find a symmetric matrix B and skew-symmetric matrix C such that $A = B + C$.

    Theorem 3.2 tells us how to do this; we take $B = (A + A^T)/2, C = (A - A^T)/2$. $B = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

5. Is A invertible? If so, find A^{-1}.

    We begin with $[A|I]$ and perform elementary row operations to put the first part into row canonical form. $\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_3+R_1 \to R_1, -4R_3+R_2 \to R_2} \begin{bmatrix} 0 & 2 & 1 & 1 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_1 \to R_1}$ $\begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 3 \\ 0 & 2 & 0 & 0 & 1 & -4 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{0.5R_2 \to R_2} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 & 0.5 & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0.5 & -2 \\ 0 & 0 & 1 & 1 & -1 & 3 \end{bmatrix}$.
    Because this was successful, A is invertible; further, $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & -2 \\ 1 & -1 & 3 \end{bmatrix}$.